Topic 11 –

Review of power series

Def: An <u>infinite sum</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots$$

where the a_i are real numbers.

We say that the sum above <u>converges</u> if there exists a real number S where

$$\lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} a_n}_{\text{these are called partial sums}} = \lim_{N\to\infty} \Big(\underbrace{a_0 + a_1 + a_2 + \cdots + a_N}_{\text{Summing the first } N \text{ terms}}\Big) = S$$

In this case we write

$$\sum_{n=0}^{\infty} a_n = S$$

If the above limit doesn't exist then we say that the infinite sum <u>diverges</u>.

Ex: Consider the sum

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

Let's calculate some partial sums

$$\sum_{n=0}^{N} \frac{1}{2^N} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^N}$$

N	$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^N}$
0	1
1	$1 + \frac{1}{2} = 1.5$
2	$1 + \frac{1}{2} + \frac{1}{2^2} = 1.75$
3	$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1.875$
4	$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 1.9375$
5	1.96875
6	1.98438
⋮	
50	$1.\underbrace{999999999999999}_{15\ 9\text{'s}}11182\dots$
⋮	
100	$1.\overbrace{999\dots99}^{30\ 9\text{'s}}211139\dots$

In Calculus you showed that this limit as $N \to \infty$ is equal to 2

and so

$$\underbrace{\sum_{n=0}^{\infty} \frac{1}{2^n} = 2}$$

in Calculus this is called a geometric sum

<u>Def:</u> A <u>power series</u> is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

Above x is a variable and the a_n and x_0 are constants. The power series is said to be <u>centered at x_0</u>.

<u>Ex:</u> (Geometric series)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\sum_{n=0}^{\infty} \underbrace{a_n}_{\substack{a_n = 1 \\ \text{for all } n}} (x - \underbrace{x_0}_{x_0 = 0})^n$$

This power series is centered at $x_0 = 0$

In Calculus you showed that

Case 1: If $-1 < x < 1$, then the geometric sum converges and

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

Case 2: If $-1 < x$ or $1 < x$, then the geometric sum

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

diverges (it doesn't have a limit)

Ex: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$$= \frac{1}{1 - \frac{1}{2}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$x = \frac{1}{2}$

$-1 < x < 1$

Ex:

$$\sum_{n=0}^{\infty} 5^n = 1 + 5 + 5^2 + 5^3 + \cdots$$

diverges

Idea:

Think of $\sum_{n=0}^{\infty} x^n$ as a function $f(x)$.

So, $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

this equals $\frac{1}{1-x}$ when $-1 < x < 1$

Then

$$f(0) = 1 + 0 + 0^2 + 0^3 + \cdots = \frac{1}{1-0} = 1$$

$$f\left(\tfrac{1}{3}\right) = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$f\left(-\tfrac{1}{4}\right) = 1 + \left(-\tfrac{1}{4}\right) + \left(-\tfrac{1}{4}\right)^2 + \left(-\tfrac{1}{4^3}\right) + \cdots$$

$$= 1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots$$

$$= \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5}$$

$f(0) = 1$

$f\left(\frac{1}{3}\right) = \frac{3}{2}$

$f\left(-\frac{1}{4}\right) = \frac{4}{5}$

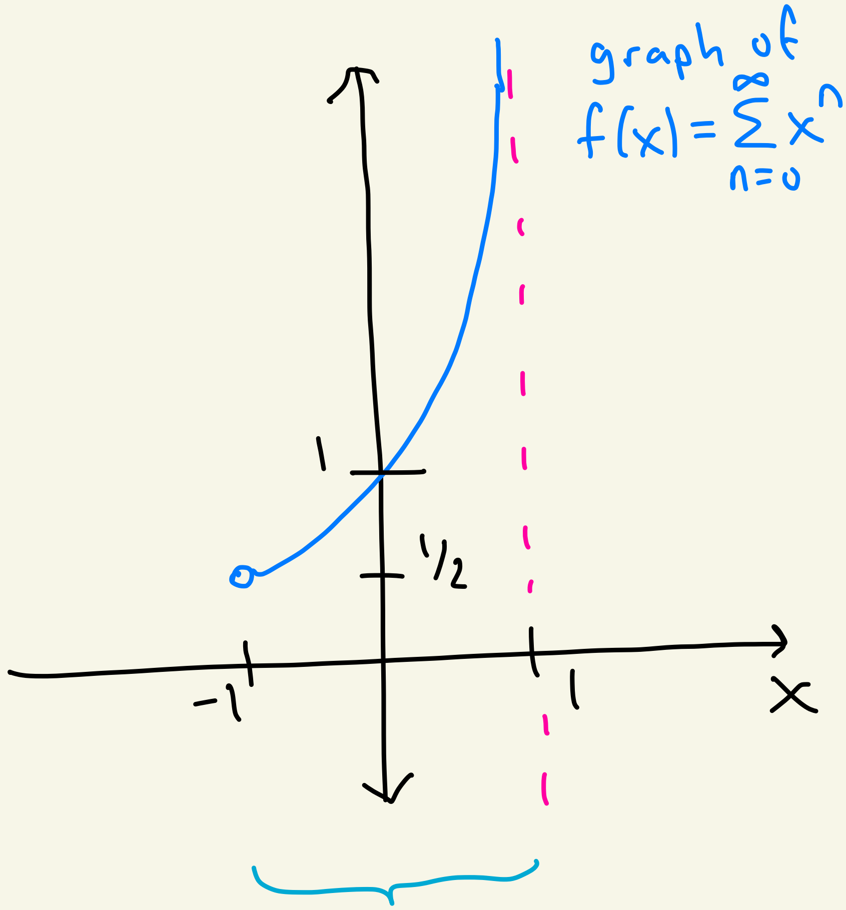
However,

$$f(2) = 1 + 2 + 2^2 + 2^3 + \cdots$$

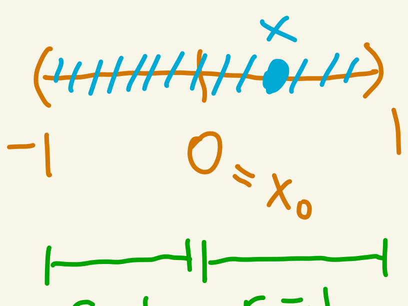
$$f(-3.2) = 1 - 3.2 + (-3.2)^2 + (-3.2)^3 + \cdots$$

are undefined.

Picture:



f is only defined in this interval

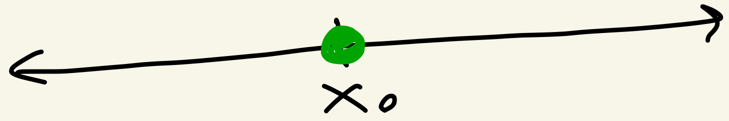


← you can only plug in x's in this range into the power series.
r=1 is called the radius of convergence

<u>Theorem:</u> There are three possible scenarios for a power series

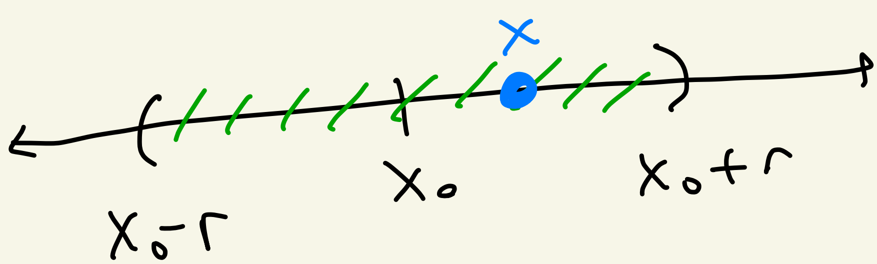
$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

① The series converges only when $x = x_0$.



Here you can only plug $x = x_0$ into the series. In this case we say the radius of convergence is $r = 0$

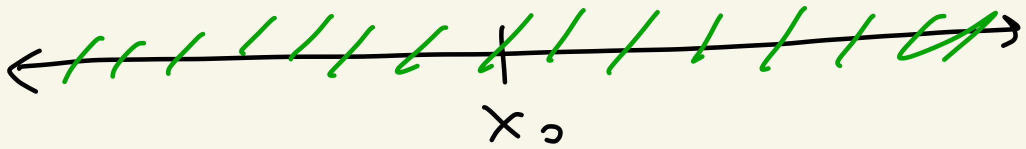
② There exists $r > 0$ where the series converges for all x when $x_0 - r < x < x_0 + r$, but it doesn't converge if $x < x_0 - r$ or $x_0 + r < x$



r is called the radius of convergence

In this case as long as x is in this interval the series converges. At past the endpoints it will diverge. At endpoints can either converge or diverge.

③ The series converges for all x.



Here $r = \infty$ is the radius of convergence.

The next examples are from Calculus.

Ex:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

converges for all x.

Here $x_0 = 0$, $r = \infty$

Ex: The sine/cosine series centered at $x_0 = 0$ are:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

These both converge for all x

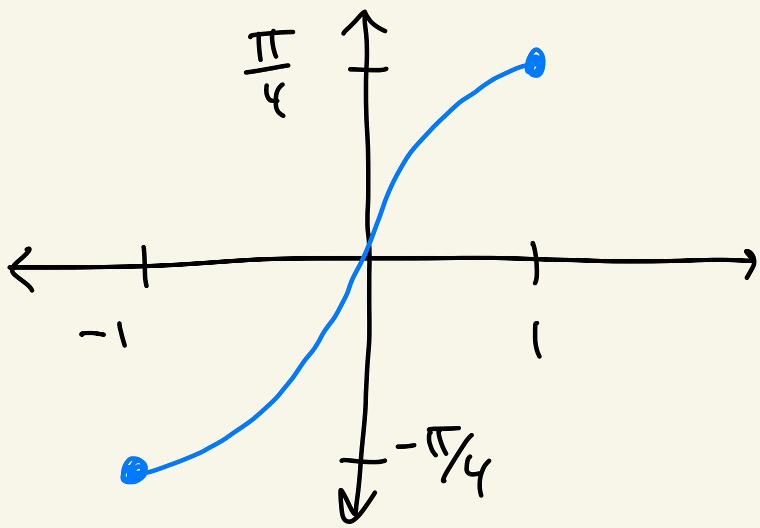
So, $r = \infty$.

Ex: If $-1 \le x \le 1$, then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

sum is only defined when $-1 \le x \le 1$



radius of convergence is $r = 1$ centered at $x_0 = 0$

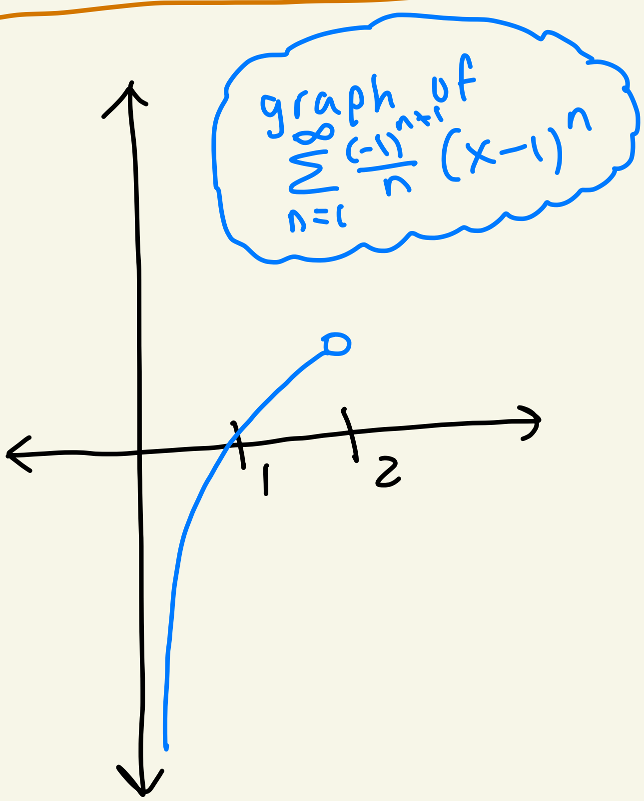
Ex: We can make a power series centered at $x_0 = 1$ that converges to $\ln(x)$. It is,

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

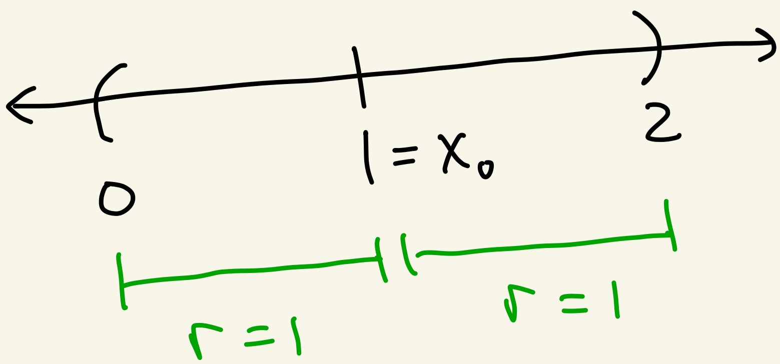
from calculus

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

It converges on when $0 < x < 2$.



Here we have:



So,

$x_0 = 1$

$r = 1$

radius of convergence

Theorem: Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

has radius of convergence $r > 0$.

Then,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Ex:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \quad \leftarrow \boxed{x_0 = 0}$$

$$= 0 + 1 \cdot x + 0 \cdot x^2 - \frac{1}{3!}x^3 + 0 \cdot x^4 - \frac{1}{5!}x^5 + \cdots$$

$f(x) = \sin(x)$
$f(0) = 0$

$f'(x) = \cos(x)$
$f'(0) = 1$
$\frac{f'(0)}{1!} = 1$

$f''(x) = -\sin(x)$
$f''(0) = 0$
$\frac{f''(0)}{2!} = 0$

$f^{(3)}(x) = -\cos(x)$
$f^{(3)}(0) = -1$
$\frac{f^{(3)}(0)}{3!} = \frac{-1}{3!}$

Ex: Find a power series for $f(x) = x^2$ centered at $x_0 = 2$.

Let's use the formula above to hopefully get an answer.

$$f(x) = x^2 \to f(2) = 4$$

$$f'(x) = 2x \to f'(2) = 4$$

$$f''(x) = 2 \to f''(2) = 2$$

$$f^{(3)}(x) = 0 \to f'''(2) = 0$$

$$\underset{k \geq 4}{f^{(k)}(x) = 0} \to \underset{k \geq 4}{f^{(k)}(2) = 0}$$

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \cdots$$

$$= 4 + 4(x-2) + \frac{2}{2}(x-2)^2 + 0(x-2)^3 + 0(x-2)^4 + \cdots$$

$$= 4 + 4(x-2) + (x-2)^2$$

One can check: $x^2 = 4 + 4(x-2) + (x-2)^2$

And the right-hand side always converges since its a finite sum. The radius of convergence is $r = \infty$, ie the formula works for all x.

Theorem: If

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

has radius of convergence $r > 0$, then

$$f'(x) = \sum_{n=1}^{\infty} n \cdot a_n (x - x_0)^{n-1}$$

$$= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots$$

and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$$

$$= a_0(x - x_0) + \frac{a_1}{2}(x - x_0)^2 + \frac{a_2}{3}(x - x_0)^3 + \cdots$$

where the power series for $f'(x)$ and $\int f(x)\,dx$ also have radii of convergence r.

Ex: Find a power series expansion for $f(x) = \frac{1}{x}$ at $x_0 = 1$.

If we only look at $x > 0$, then

$$\frac{1}{x} = \frac{d}{dx} \ln(x)$$

$$= \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \right]$$

$\boxed{0 < x < 2}$

$$= \frac{d}{dx}\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots \right]$$

$$= 1 - (x-1) + (x-1)^2 - \dots$$

So,

$$\frac{1}{x} = \sum_{n=1}^{\infty} (-1)^{n+1} (x-1)^{n-1} = 1 - (x-1) + (x-1)^2 - \dots$$

which as radius of convergence $r = 1$ about $x_0 = 1$.

So the series converges for $0 < x < 2$